

REGULARITY OF STATIONARY SOLUTIONS TO THE LINEARIZED BOLTZMANN EQUATIONS

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ABSTRACT. We consider the regularity of stationary solutions to the linearized Boltzmann equations in bounded C^1 convex domains in \mathbb{R}^3 for gases with cutoff hard potential and cutoff Maxwellian gases. We prove that the stationary solutions are Hölder continuous with order $\frac{1}{2}^-$ away from the boundary provided the incoming data have the same regularity. The key idea is to partially transfer the regularity in velocity obtained by collision to space through transport and collision.

1. INTRODUCTION

A thermal non-equilibrium stationary solution, a resolution between the thermal dynamical tendency and boundary effects, reflects the complexity and richness of the Boltzmann equations. At linear and weakly nonlinear levels, the existence of such solutions has been studied by Guiraud [10, 11] for convex domains and by Esposito, Guo, Kim, and Marra [6] for nonconvex domains in \mathbb{R}^3 . Regarding the regularity issue, as far as we know, the best result is that the solution is continuous away from the grazing set also in [6].

For the corresponding time evolutionary weakly nonlinear problems, it was studied by Kim, [14], that the discontinuity propagates from the boundary to the interior of a non-convex domain on the tangent direction. The BV property of solutions in a non-convex domain was studied by Guo, Kim, Tonon, and Trescases [13]. By the same authors, regularity problem for convex domains was studied in [12]. Especially, they established the weighted C^1 estimate for the specular reflection and diffuse reflection boundary conditions. However, the weighted C^1 norm grows severely with time, and therefore, could not give information to the stationary solution through large time behavior. This motivates us to look at the regularity to the stationary solution directly.

The goal of this paper is to prove the interior regularity of the stationary linearized Boltzmann equation provided that the incoming data have the same regularity. After first laying out the problem setting and definitions, the precise statement of the main theorem will follow.

We consider the stationary linearized Boltzmann equation

$$(1.1) \quad \zeta \cdot \nabla f(x, \zeta) = L(f),$$

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where $\zeta \in \mathbb{R}^3$ and $x \in \Omega$, a C^1 bounded convex domain in \mathbb{R}^3 . The linear collision operator L here is corresponding to the cutoff hard potential gases or cutoff Maxwellian gases, which are indicated by $0 \leq \gamma < 1$ (see (1.2)). This cutoff was first introduced by Grad [9], and later the analysis was refined by Caflisch [1]. Furthermore, we assume the cross section is a product of a function of the length of relative velocity and a function of the deflecting angle as in [4], i.e., we assume that the cross section is nontrivial and satisfies

$$(1.2) \quad B(|\zeta_* - \zeta|, \theta) = |\zeta_* - \zeta|^\gamma \beta(\theta), \quad 0 \leq \beta(\theta) \leq C \cos \theta \sin \theta.$$

Here, the cross section is for the binary collision operator

$$(1.3) \quad Q(F, F) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (F' F'_* - F F_*) B(|\zeta_* - \zeta|, \theta) d\theta d\epsilon d\xi_*$$

before linearized around the standard Maxwellian $\pi^{-\frac{3}{2}} e^{-|\zeta|^2}$. Under this assumption, L has the following known properties (See [1, 4, 9]). L can be decomposed into a multiplication operator and an integral operator:

$$(1.4) \quad L(f) = -\nu(|\zeta|)f + K(f),$$

where

$$(1.5) \quad K(f)(x, \zeta) = \int_{\mathbb{R}^3} k(\zeta, \zeta_*) f(x, \zeta_*) d\zeta_*$$

is symmetric. The explicit expression of ν is

$$(1.6) \quad \nu(|\zeta|) = \beta_0 \int_{\mathbb{R}^3} e^{-|\eta|^2} |\eta - \zeta|^\gamma d\eta,$$

where $\beta_0 = \int_0^{\frac{\pi}{2}} \beta(\theta) d\theta$. Let $0 < \delta < 1$. The collision frequency $\nu(|\zeta|)$ and the collision kernel $k(\zeta, \zeta_*)$ satisfies

$$(1.7)$$

$$\nu_0(1 + |\zeta|)^\gamma \leq \nu(|\zeta|) \leq \nu_1(1 + |\zeta|)^\gamma,$$

$$(1.8)$$

$$|k(\zeta, \zeta_*)| \leq C_1 |\zeta - \zeta_*|^{-1} (1 + |\zeta| + |\zeta_*|)^{-(1-\gamma)} e^{-\frac{1-\delta}{4} \left(|\zeta - \zeta_*|^2 + \left(\frac{|\zeta|^2 - |\zeta_*|^2}{|\zeta - \zeta_*|} \right)^2 \right)},$$

$$(1.9)$$

$$|\nabla_\zeta k(\zeta, \zeta_*)| \leq C_2 \frac{1 + |\zeta|}{|\zeta - \zeta_*|^2} (1 + |\zeta| + |\zeta_*|)^{-(1-\gamma)} e^{-\frac{1-\delta}{4} \left(|\zeta - \zeta_*|^2 + \left(\frac{|\zeta|^2 - |\zeta_*|^2}{|\zeta - \zeta_*|} \right)^2 \right)}.$$

Here, the constants $0 < \nu_0 < \nu_1$ may depend on the potential and C_1 and C_2 may depend on δ and the potential. Related to the above estimates, the following proposition from [1] is crucial in our study.

Proposition 1.1. *For any $\epsilon, a_1, a_2 > 0$,*

$$(1.10) \quad \left| \int_{\mathbb{R}^3} \frac{1}{|\eta - \zeta_*|^{3-\epsilon}} e^{-a_1|\eta - \zeta_*|^2 - a_2 \frac{(|\eta|^2 - |\zeta_*|^2)^2}{|\eta - \zeta_*|^2}} d\zeta_* \right| \leq C_4(1 + |\eta|)^{-1},$$

where C_4 may depend on ϵ, a_1 , and a_2 .

Suppose x is a point inside Ω . We define $p(x, \zeta)$ to be the boundary point that the backward trajectory from x with velocity ζ touches. The corresponding traveling time is denoted by $\tau_-(x, \zeta)$. We view the integral operator K as the source term:

$$(1.11) \quad \zeta \nabla f(x, \zeta) + \nu(|\zeta|)f(x, \zeta) = K(f).$$

Using the method of characteristics, we derive the corresponding corresponding integral equation:

$$(1.12) \quad f(x, \zeta) = f(p(x, \zeta), \zeta) e^{-\nu(|\zeta|)\tau_-(x, \zeta)} + \int_0^{\tau_-(x, \zeta)} e^{-\nu(|\zeta|)s} K(f)(x - \zeta s, \zeta) ds.$$

In this paper, we say f is a solution to (1.1) if f satisfies (1.12) almost everywhere.

The boundary of Ω is denoted by $\partial\Omega$, and the outer normal is denoted by \vec{n} . We define

$$(1.13) \quad \Gamma_- := \{(x, \zeta) | x \in \partial\Omega, \zeta \cdot \vec{n}(x) < 0\}.$$

We consider norms as follows:

$$(1.14) \quad \|g(\zeta)\|_{L_\zeta^*} := \left(\int_{\mathbb{R}^3} |g(\zeta)|^2 \nu(|\zeta|) d\zeta \right)^{\frac{1}{2}},$$

$$(1.15) \quad \|f(x, \zeta)\|_{L_{x, \zeta}^*} := \left(\int_{\Omega} \int_{\mathbb{R}^3} |f(x, \zeta)|^2 \nu(|\zeta|) d\zeta dx \right)^{\frac{1}{2}},$$

$$(1.16) \quad \|f(x, \zeta)\|_{L_x^\infty L_\zeta^*} := \sup_{x \in \Omega} \left(\int_{\mathbb{R}^3} |f(x, \zeta)|^2 \nu(|\zeta|) d\zeta \right)^{\frac{1}{2}},$$

$$(1.17) \quad \|f(x, \zeta)\|_{L_{x, \zeta}^\infty} := \sup_{x \in \Omega, \zeta \in \mathbb{R}^3} |f(x, \zeta)|.$$

The indices above denote the corresponding functional spaces.

The main conclusion in this paper is as follows.

Theorem 1.2. *Let $0 < \sigma < \frac{1}{2}$ and $f \in L_{x, \zeta}^*$ be a stationary solution to the linearized Boltzmann equations (1.1),*

$$\zeta \cdot \nabla f(x, \zeta) = L(f),$$

in a C^1 bounded convex domain Ω in \mathbb{R}^3 for gases with cutoff hard potential or cutoff Maxwellian gases, i.e., the corresponding cross sections satisfy (1.2) for $0 \leq \gamma < 1$. Suppose there exist $\phi(\zeta) \in L_\zeta^$ and $M > 0$ such that*

$$(1.18) \quad |f(X, \eta)| \leq \phi(\eta),$$

$$(1.19) \quad |f(X, \eta) - f(Y, \omega)| \leq M (|\eta - \omega|^2 + |X - Y|^2)^{\frac{\sigma}{2}}$$

for any $(X, \eta), (Y, \omega) \in \Gamma_-$.

Then, $f \in L_{x, \zeta}^\infty$. Furthermore, there exists a constant C_0 depending only on $\|f\|_{L_{x, \zeta}^\infty}$, σ , M , Ω , and the potential such that, for any $x, y \in \Omega$ and $\zeta, \xi \in \mathbb{R}^3$,

$$(1.20) \quad |f(x, \zeta) - f(y, \xi)| \leq C_0(1 + d_0^{-1})^3 (|\zeta - \xi|^2 + |x - y|^2)^{\frac{\sigma}{2}},$$

where d_0 is the distance of x, y to $\partial\Omega$.

Notice that $L_{x, \zeta}^*$ is the very functional space for the existence results in [6, 10]. Also, notice that the inequality (1.20) breaks down at boundary, while some observations from the explicit simple example of the thermal transport problem may suggest so (see [3, 5, 18]). Although the hard sphere case is barely missed by the theorem, if we know solution is bounded a priori, we can recover the theorem for $\gamma = 1$.

Our strategy is as follows. We break the theorem down into three parts: $L_{x, \zeta}^* \rightarrow L_x^\infty L_\zeta^*$, $L_x^\infty L_\zeta^* \rightarrow L_{x, \zeta}^\infty$, and $L_{x, \zeta}^\infty$ to Hölder continuity.

In one space dimensional case, Golse and Poupaud used a bootstrap strategy to boost the integrability of solutions from $L_{x, \zeta}^*$ to $L_x^\infty L_\zeta^*$ in finite steps for Milne and Kramers' problems, [8]. The key ingredient is to convert an estimate in velocity to a convolution in space, thanks to the transport and collision nature of Boltzmann equations and the simple geometry of one space dimension. For the three dimensional problem, we manage to overcome the obstacle in geometry and extend the method to obtain $L_x^\infty L_\zeta^*$ regularity. Then, together with the assumption of Hölder continuity for the incoming data, we can show the solution is a bounded function in Section 5.

It is well known that the integral operator K can improve the regularity in velocity. For the regularity in space, the Hölder continuity of $K(f)$ in space for solutions to one space dimensional equations is proved in [2] for hard sphere gases, again thanks to the simple geometry of one space dimension. This observation also plays an important role in [4]. More precisely, in one space dimensional case, any two points can be connected by a trajectory, either forward or backward, for almost all velocity. This is certainly not true in a three dimensional space.

To carry out the Hölder continuity for the three space dimensional problem we consider, we first iterate the integral equation once more:

$$(1.21) \quad \begin{aligned} f(x, \zeta) &= f(p(x, \zeta), \zeta) e^{-\nu(|\zeta|)\tau_-(x, \zeta)} \\ &+ \int_0^{\tau_-(x, \zeta)} \int_{\mathbb{R}^3} e^{-\nu(|\zeta|)s} k(\zeta, \zeta') e^{-\nu(|\zeta'|)\tau_-(x - \zeta s, \zeta')} f(p(x - \zeta s, \zeta'), \zeta') d\zeta' ds \\ &+ \int_0^{\tau_-(x, \zeta)} \int_{\mathbb{R}^3} \int_0^{\tau_-(x - \zeta s, \zeta')} e^{-\nu(|\zeta|)s} k(\zeta, \zeta') e^{-\nu(|\zeta'|)t} K(f)(x - \zeta s - \zeta' t, \zeta') dt d\zeta' ds \\ &=: I + II + III. \end{aligned}$$

The I and II can preserve the regularity of Γ_- due to the nature of the transport equation. The key step is to prove the increasing of regularity in III .

The smoothing effect of K in velocity has been studied, for example, in [2, 4, 16]. The key observation in this study is that, in III , the regularity in velocity of $K(f)$ can be partially transferred to space due to transport and collision. In order to connect velocity to space, in Section 2, we change coordinates in the inner integral in III to space over Ω and length of velocity over \mathbb{R}^+ . Due to singularity of the integrand in that formula, we are not able to push the regularity in space to differentiable. However, using the fact that angle difference from an observer toward two different points, the parallax, becomes smaller when the observer becomes farer away and balance the contribution from nearby by the smallness of domain of integration, we are able to obtain local Hölder continuity of III to the order of $\frac{1}{2}$.

Readers familiar to the time evolutionary Boltzmann equations in the whole Euclidean space may find an analogy to the Mixture Lemma studied by Liu and Yu [16, 17], and later extend by Kuo, Liu, and Noh [15], and Wu [19], that is, the regularity in velocity can be transfer to space through collision and transport. This kind of idea can be traced back to the famous Averaging Lemma by Golse, Perthame, and Sentis [7]. In the context of stationary solution in a convex domain, despite of the same spirit, the new technique to carry out the mixture effect as explained above is quite different.

The plan of this paper is as follows. In Section 2, we study the key mixture effect. Then, we investigate the geometric properties of a convex domain in Section 3, where we show the local Hölder continuity of III . In section 4, we show that the regularity of the boundary data are preserved by first and second iterations of the transport equation, I and II . Eventually, we return to increasing of integrability from $L_{x,\zeta}^*$ to $L_{x,\zeta}^\infty$ in section 5 and complete our proof.

2. GAINING REGULARITY FROM COLLISION AND TRANSPORT

We will elaborate the smoothing effect due to the combination of collision and transport in this section. We first convert the formula by interplaying between velocity and space. We change ζ' to the spherical coordinates so that

$$(2.1) \quad \zeta' = (\rho \cos \theta, \rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi).$$

Also, we change traveling time to the traveling distance:

$$(2.2) \quad r = \rho t.$$

Let $\hat{\zeta}' = \frac{\zeta'}{|\zeta'|}$. Then,

$$\begin{aligned}
(2.3) \quad III &= \int_0^{\tau_-(x, \zeta)} e^{-\nu(|\zeta|)s} \int_0^\infty \int_0^\pi \int_0^{2\pi} \int_0^{|\overline{xp(x-s\zeta, \zeta')}|} \\
&\quad k(\zeta, \zeta') e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x - \zeta s - \hat{\zeta}'r, \zeta') \rho \sin \theta dr d\phi d\theta dp ds \\
&= : \int_0^{\tau_-(x, \zeta)} e^{-\nu(|\zeta|)s} G(x - \zeta s, \zeta) ds.
\end{aligned}$$

Notice that we can parametrize Ω by θ , ϕ , and r , thanks to the convexity. Therefore, by regrouping the integrals, we can change the formulation to contain an integral over space. Let $x_0 = x - \zeta s$ and $y = x - \zeta s - \hat{\zeta}'r$. We have

$$\begin{aligned}
(2.4) \quad G(x_0, \zeta) &= \\
&\int_0^\infty \int_\Omega k\left(\zeta, \frac{(x_0 - y)\rho}{|x_0 - y|}\right) e^{-\nu(\rho)\frac{|x_0 - y|}{\rho}} K(f)\left(y, \frac{(x_0 - y)\rho}{|x_0 - y|}\right) \frac{\rho}{|x_0 - y|^2} dy d\rho.
\end{aligned}$$

Notice that if we differentiate the above formula directly respect to x_0 , it will become not integrable in y . However, we can still obtain a lower regularity from this expression. The main purpose of this section is to prove the following lemma.

Lemma 2.1. *Suppose $f(x, \zeta) \in L_{x, \zeta}^\infty$ is a solution to (1.1) and $x_0, x_1 \in \Omega$. Then, there exist C_3 only depending on the Ω and the potential such that*

$$(2.5) \quad |G(x_0, \zeta) - G(x_1, \zeta)| \leq C_3 \|f\|_{L_{x, \zeta}^\infty} |x_0 - x_1|^{\frac{1}{2}}.$$

We will follow the strategy explained in the introduction. As mentioned, the smoothing effect of K in velocity is a well known fact. It was used to study Green's functions of the Boltzmann equations in [16]. In [2], the locally Hölder continuity of $K(f)$ in velocity has been shown for $f \in L_\zeta^2$ for hard sphere gases. Here, we will use an estimate appeared in [4].

Proposition 2.2. *Suppose $1 \leq p \leq \infty$ and $f \in L_\zeta^p$. Then, there exists C_p depending only on p and the potential such that*

$$(2.6) \quad \|\nabla_\zeta K(f)(\zeta)\|_{L_\zeta^p} \leq C_p \|f\|_{L_\zeta^p}.$$

Using (1.9) and Proposition 1.1 we can conclude that both $\|\nabla_\zeta k(\zeta, \zeta_*)\|_{L_\zeta^\infty L_{\zeta_*}^1}$ and $\|\nabla_\zeta k(\zeta, \zeta_*)\|_{L_{\zeta_*}^\infty L_\zeta^1}$ are bounded. Therefore, we can prove the Proposition 2.2 by an argument similar to the proof of Young's inequality.

In addition to the above formula, we also need an estimate for the parallax to transfer the regularity from velocity to space.

Proof of Lemma 2.1. We add and subtract to obtain

(2.12)

$$\begin{aligned}
|G(x_0, \zeta) - G(x_1, \zeta)| &\leq \left| \int_0^\infty \int_\Omega k(\zeta, \frac{(x_0 - y)\rho}{|x_0 - y|}) \frac{\rho e^{-\nu(\rho) \frac{|x_0 - y|}{\rho}}}{|x_0 - y|^2} \right. \\
&\quad \cdot \left[K(f)(y, \frac{(x_0 - y)\rho}{|x_0 - y|}) - K(f)(y, \frac{(x_1 - y)\rho}{|x_1 - y|}) \right] dy d\rho \Big| \\
&+ \left| \int_0^\infty \int_\Omega K(f)(y, \frac{(x_1 - y)\rho}{|x_1 - y|}) \right. \\
&\quad \cdot \left[k(\zeta, \frac{(x_0 - y)\rho}{|x_0 - y|}) \frac{\rho e^{-\nu(\rho) \frac{|x_0 - y|}{\rho}}}{|x_0 - y|^2} - k(\zeta, \frac{(x_1 - y)\rho}{|x_1 - y|}) \frac{\rho e^{-\nu(\rho) \frac{|x_1 - y|}{\rho}}}{|x_1 - y|^2} \right] dy d\rho \Big| \\
&=: G_K + G_O.
\end{aligned}$$

We first deal with G_K . We use $B(x, r)$ to denote the open ball centered at x with radius r . We break the domain of integration into two, $\Omega_1 := \Omega \setminus B(x_0, 2d^a)$ and $\Omega_2 := \Omega \cap B(x_0, 2d^a)$ for $0 < a < 1$ to be determinate later and name the corresponding integrals as A_1 and A_2 respectively. For A_1 , we will use the smoothness of $K(f)$ in velocity, while, for A_2 , we use the smallness of domain. Notice that, from Proposition 2.2, $K(f)$ is Lipschitz continuous in velocity.

$$(2.13) \quad |K(f)(y, \zeta_1) - K(f)(y, \zeta_2)| \leq C \|f\|_{L_{x, \zeta}^\infty} |\zeta_1 - \zeta_2|.$$

Together with Proposition 2.3, we have

$$\begin{aligned}
&\left| K(f)(y, \frac{(x_0 - y)\rho}{|x_0 - y|}) - K(f)(y, \frac{(x_1 - y)\rho}{|x_1 - y|}) \right| \\
(2.14) \quad &\leq C \|f\|_{L_{x, \zeta}^\infty} \left| \frac{(x_0 - y)\rho}{|x_0 - y|} - \frac{(x_1 - y)\rho}{|x_1 - y|} \right| \leq C \|f\|_{L_{x, \zeta}^\infty} \rho \theta \\
&\leq C \|f\|_{L_{x, \zeta}^\infty} \rho d^{1-a}.
\end{aligned}$$

Therefore,

(2.15)

$$\begin{aligned}
|A_1| &\leq C \|f\|_{L_{x, \zeta}^\infty} \int_0^\infty \int_{\Omega_1} \left| k(\zeta, \frac{(x_0 - y)\rho}{|x_0 - y|}) \right| \frac{\rho e^{-\nu(\rho) \frac{|x_0 - y|}{\rho}}}{|x_0 - y|^2} d^{(1-a)} \rho dy d\rho \\
&\leq C d^{(1-a)} \|f\|_{L_{x, \zeta}^\infty} \int_0^\infty \int_0^\pi \int_0^{2\pi} \int_{2d^a}^R \left| k(\zeta, \frac{(x_0 - y)\rho}{|x_0 - y|}) \right| \rho^2 \sin \theta dr d\phi d\theta d\rho \\
&\leq C d^{(1-a)} \|f\|_{L_{x, \zeta}^\infty} \int_{2d^a}^R \int_{\mathbb{R}^3} |k(\zeta, \zeta')| d\zeta' dr \\
&\leq C d^{(1-a)} \|f\|_{L_{x, \zeta}^\infty} R,
\end{aligned}$$

where R is the diameter of Ω . Notice that we changed back the coordinates in order to take advantage of the integrability of $k(\zeta, \zeta')$ in the above inequality.

On the other hand, using (1.8), we have

$$(2.16) \quad \|K(f)(x, \zeta)\|_{L_{x, \zeta}^\infty} \leq C\|f\|_{L_{x, \zeta}^\infty}.$$

Therefore,

$$(2.17) \quad \begin{aligned} |A_2| &\leq C\|f\|_{L_{x, \zeta}^\infty} \int_0^\infty \int_{B(x_0, 2d^a)} \left| k\left(\zeta, \frac{(x_0 - y)\rho}{|x_0 - y|}\right) \right| e^{-\nu(\rho) \frac{|x_0 - y|}{\rho}} \frac{\rho}{|x_0 - y|^2} dy d\rho \\ &\leq C\|f\|_{L_{x, \zeta}^\infty} \int_0^\infty \int_0^\pi \int_0^{2\pi} \int_0^{2d^a} \left| k\left(\zeta, \frac{(x_0 - y)\rho}{|x_0 - y|}\right) \right| e^{-\nu(\rho) \frac{|x_0 - y|}{\rho}} \rho \sin \theta dr d\phi d\theta d\rho \\ &\leq C\|f\|_{L_{x, \zeta}^\infty} \int_0^{2d^a} \int_{\mathbb{R}^3} |k(\zeta, \zeta')| \frac{1}{|\zeta'|} d\zeta' dr \leq C\|f\|_{L_{x, \zeta}^\infty} d^a. \end{aligned}$$

To optimize the estimate, we choose $a = \frac{1}{2}$, which gives the desired estimate for G_K .

Now, we proceed to estimate G_O . We divide the domain of integration into two, $\Omega_3 := \Omega \setminus B(x_0, 2d)$ and $\Omega_4 := \Omega \cap B(x_0, 2d)$, and name the corresponding integrals as A_3 and A_4 respectively.

We first deal with A_3 . Let $\tilde{x}(u) = x_1 + (x_0 - x_1)u$. We have

$$(2.18) \quad |A_3| = \left| \int_0^\infty \int_{\Omega_3} \int_0^1 \frac{d}{du} \left[k\left(\zeta, \frac{(\tilde{x}(u) - y)\rho}{|\tilde{x}(u) - y|}\right) \frac{\rho e^{-\nu(\rho) \frac{|\tilde{x}(u) - y|}{\rho}}}{|\tilde{x}(u) - y|^2} \right] du K(f)(y, \frac{(x_1 - y)\rho}{|x_1 - y|}) dy d\rho \right|.$$

By direct calculation and using (1.8), (1.9), (2.16), together with Proposition 1.1, we obtain

$$\begin{aligned}
(2.19) \quad & |A_3| \\
& \leq C|x_1 - x_0| \|f\|_{L_{x,\zeta}^\infty} \int_0^1 \int_0^\infty \int_{\Omega_3} e^{-\frac{1}{8} \left[|\zeta - \frac{(\tilde{x}(u)-y)\rho}{|\tilde{x}(u)-y|}|^2 + (\frac{|\zeta|^2 - \rho^2}{|\zeta - \zeta'(\rho, \tilde{x}(u), y)|})^2 \right]} \\
& \quad \cdot \frac{1}{|\tilde{x}(u) - y|^3} \left[\frac{\rho^2(1 + \rho)}{|\zeta - \frac{(\tilde{x}(u)-y)\rho}{|\tilde{x}(u)-y|}|^2} + \frac{\rho}{|\zeta - \frac{(\tilde{x}(u)-y)\rho}{|\tilde{x}(u)-y|}|} \right] dy dp du \\
& \leq Cd \|f\|_{L_{x,\zeta}^\infty} \int_0^1 \int_d^R \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{e^{-\frac{1}{8} \left[|\zeta - \frac{(\tilde{x}(u)-y)\rho}{|\tilde{x}(u)-y|}|^2 + (\frac{|\zeta|^2 - \rho^2}{|\zeta - \zeta'(\rho, \tilde{x}(u), y)|})^2 \right]}}{r} \\
& \quad \cdot \left[\frac{\rho^2(1 + \rho)}{|\zeta - \frac{(\tilde{x}(u)-y)\rho}{|\tilde{x}(u)-y|}|^2} + \frac{\rho}{|\zeta - \frac{(\tilde{x}(u)-y)\rho}{|\tilde{x}(u)-y|}|} \right] \sin \theta d\rho d\phi d\theta dr du \\
& \leq Cd \|f\|_{L_{x,\zeta}^\infty} \int_0^1 \int_d^R \int_{\mathbb{R}^3} e^{-\frac{1}{8} \left[|\zeta - \zeta'|^2 + (\frac{|\zeta|^2 - |\zeta'|^2}{|\zeta - \zeta'|})^2 \right]} \\
& \quad \cdot \left[\frac{1 + |\zeta| + |\zeta - \zeta'|}{|\zeta - \zeta'|^2} + \frac{1}{|\zeta - \zeta'| |\zeta'|} \right] d\zeta' \frac{1}{r} dr du \\
& \leq Cd(1 + |\ln d|) \|f\|_{L_{x,\zeta}^\infty}.
\end{aligned}$$

Notice that we applied

$$(2.20) \quad |\tilde{x}(u) - y| \geq |x_0 - y| - |\tilde{x}(u) - x_0| \geq d$$

in the above inequality.

On the other hand, notice that $\Omega_4 \subseteq B(x_0, 2d) \subseteq B(x_1, 3d)$. We can obtain

$$(2.21) \quad |A_4| \leq Cd \|f\|_{L_{x,\zeta}^\infty}.$$

Combining all the estimates above, we conclude the lemma. \square

In the end of this section, we will discuss the regularity of G in velocity. Notice that

$$(2.22) \quad \left| \int_0^{\tau_-(x_0, \zeta')} e^{-\nu t} K(f)(x_0 - \zeta' t, \zeta') dt \right| < C \|f\|_{L_{x,\zeta}^\infty}.$$

Applying Proposition 2.2, we can conclude

Proposition 2.4.

$$(2.23) \quad |G(x_0, \zeta_1) - G(x_0, \zeta_2)| \leq C_5 \|f\|_{L_{x,\zeta}^\infty} |\zeta_1 - \zeta_2|,$$

where the constant C_5 depending only on the potential.

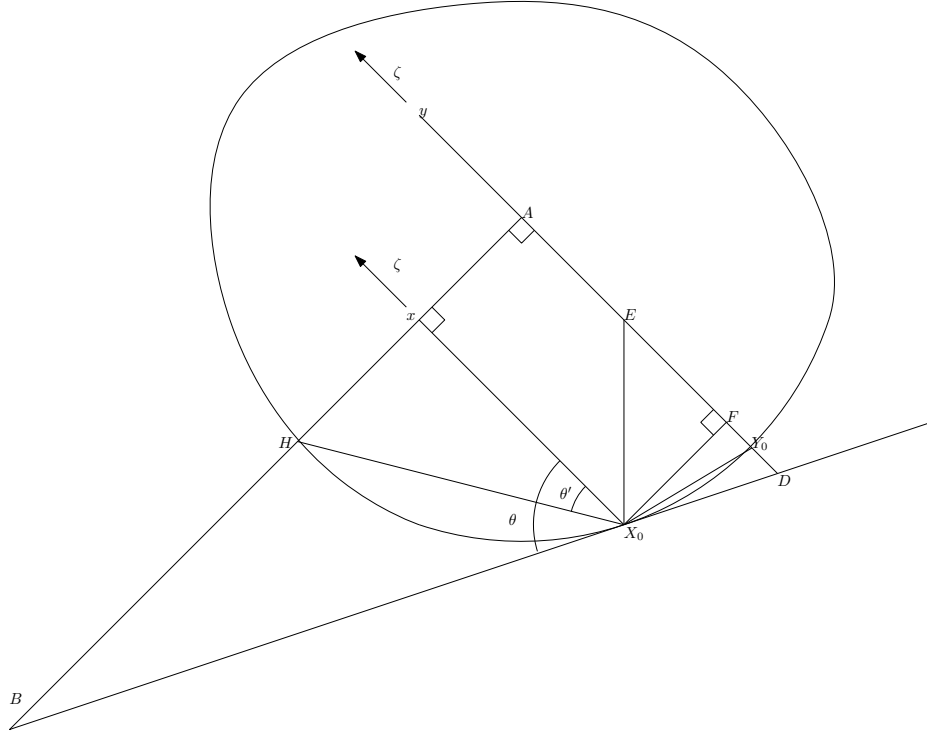


FIGURE 2.

3. PROPERTIES OF A CONVEX DOMAIN

In this section, we will discuss some properties of a convex domain and then apply to the proof of the regularity of *III*. These properties are all based on the same observation, that is, a line connecting an interior point and a boundary point can not have too small an angle to the tangent plane in a convex domain if the point is away from the boundary. The following proposition and its proof serve as an examples.

Proposition 3.1. *Suppose Ω is a bounded C^1 convex domain in \mathbb{R}^3 and $x, y \in \Omega$. Then, there exists a constant C_6 only depending on Ω such that*

$$(3.1) \quad |p(x, \zeta) - p(y, \zeta)| \leq C_6(1 + d_0^{-1})|x - y|,$$

$$(3.2) \quad |\tau_-(x, \zeta) - \tau_-(y, \zeta)| \leq C_6(1 + d_0^{-1}) \frac{|x - y|}{|\zeta|}.$$

Proof. We will work on the geometry of the plane passing two points x and y and containing the vector ζ as showing in Figure 2. Without loss of generality, we may assume $\tau_-(y, \zeta) \geq \tau_-(x, \zeta)$. We note $X_0 = p(x, \zeta)$ and $Y_0 = p(y, \zeta)$. If $y - x \parallel \zeta$, then $X_0 = Y_0$ and $|\tau_-(x, \zeta) - \tau_-(y, \zeta)| = \frac{|x - y|}{|\zeta|}$ and the lemma holds. Otherwise, let ζ_\perp be the projection of the vector $x - y$ on the subspace perpendicular to ζ . The line passing x with direction ζ_\perp

is denoted by L_\perp , which intersects the line $\overleftrightarrow{yY_0}$ at a point A . Suppose L_\perp intersects the tangent plane of Ω passing X_0 at a point B . Then, the line segment $\overline{x\overline{B}}$ intersects $\partial\Omega$ at a point H , because of the convexity. $\overleftrightarrow{B\overline{X_0}}$ and $\overleftrightarrow{yY_0}$ intersect at D . Let $E = X_0 + (y - x)$ and $F = X_0 + (A - x)$. Define $\theta := \angle ADB = \angle xX_0B$. Notice that $0 < \theta \leq \frac{\pi}{2}$ because $\overleftrightarrow{Ax} \perp \overleftrightarrow{AD}$. We observe $0 < \theta' := \angle xX_0H < \theta$ because of the convexity. Notice that

$$(3.3) \quad \sin \theta' = \frac{|\overline{xH}|}{|\overline{X_0H}|} \geq \frac{d_0}{R}.$$

Therefore,

$$(3.4) \quad |\overline{X_0D}| = \frac{|\overline{X_0F}|}{\sin \theta} \leq \frac{R}{d_0} |\overline{X_0F}| \leq \frac{R}{d_0} |x - y|.$$

Notice that $|\zeta| \cdot |\tau_-(x, \zeta) - \tau_-(y, \zeta)| = |\overline{EY_0}| < |\overline{ED}|$. Then, applying triangular inequality for $\triangle EDX_0$, we can conclude (3.2). Again, using triangular inequality for $\triangle EY_0X_0$, we obtain (3.1). Therefore, we proved the proposition for the case L_\perp intersects with the tangent plane passing x . For the other case, i.e., if L_\perp is parallel to the tangent plane passing X_0 , we can easily see that the proposition holds. \square

We use $d(z, \partial\Omega)$ to denote the distance from z to $\partial\Omega$. By a similar argument, we have the following Proposition to be used the next section

Proposition 3.2. *With the same assumption in Proposition 3.1. Suppose $z \in \overline{xX_0}$. Then,*

$$(3.5) \quad |\overline{zX}| \leq \frac{R}{d_0} d(z, \partial\Omega).$$

For two trajectories passing the same point, we have the following proposition.

Proposition 3.3. *Suppose Ω is a bounded C^1 convex domain and $\zeta_1, \zeta_2 \in \mathbb{R}^3$ and $x \in \Omega$. Let $P_1 = p(x, \zeta_1)$ and $P_2 = p(x, \zeta_2)$. Let d_0 be the distance between x and $\partial\Omega$ and θ be the angle between ζ_1 and ζ_2 . Then, there exists a constant C_7 depending only on Ω such that*

$$(3.6) \quad |P_1 - P_2| \leq C_7(1 + d_0^{-1})\theta,$$

$$(3.7) \quad |\overline{xP_1}| - |\overline{xP_2}| \leq C_7(1 + d_0^{-1})\theta.$$

Proof. When θ is big, the lemma is true because Ω is bounded. Thus, we only need to deal with the case $\theta < 1$. We are going to work on the plane geometry of the plane passing x , P_1 , and P_2 . Without loss of generality, we can assume $|\overline{xP_2}| \geq |\overline{xP_1}|$. Let A' be the projection of P_1 on $\overline{xP_2}$. Notice that

$$(3.8) \quad |\overline{P_1A'}| \leq R \sin \theta \leq R\theta.$$

On the plane we considered, the line passing x and perpendicular to $\overline{xP_2}$ intersects Ω at two points. The one at the same side with P_1 is denoted by B' . Because of the convexity, we observe

$$(3.9) \quad \frac{d_0}{R} \leq \sin \angle B'P_2x \leq \sin \angle P_1P_2x.$$

Therefore,

$$(3.10) \quad |\overline{P_1P_2}| = \frac{|\overline{P_1A'}|}{\sin \angle P_1P_2x} \leq \frac{R^2}{d_0}\theta.$$

Also,

$$(3.11) \quad |\overline{xP_1}| - |\overline{xP_2}| \leq |\overline{A'P_2}| \leq |\overline{P_1P_2}| \leq \frac{R^2}{d_0}\theta.$$

Hence, we complete the proof. \square

Now we are ready to discuss the regularity of III .

Proposition 3.4. *Let Ω be a bounded C^1 domain in \mathbb{R}^3 . III is defined in (1.21) for $f \in L_{x,\zeta}^\infty$ that solves (1.1). Suppose x and y are interior points of Ω and $\zeta_1, \zeta_2 \in \mathbb{R}^3$. Then, there exists C_8 depending only on Ω and the potential of the gases, such that*

$$(3.12) \quad |III(x, \zeta_1) - III(y, \zeta_2)| \leq C_8(1 + d_0^{-1})^3 \|f\|_{L_{x,\zeta}^\infty} (|x - y|^2 + |\zeta_1 - \zeta_2|^2)^{\frac{1}{4}},$$

where d_0 is the distance of x and y to $\partial\Omega$.

Proof. Since III is bounded, the cases for $|x - y| > 1$ or $|\zeta_1 - \zeta_2| > 1$ are trivial. We only consider the case $|x - y| \leq 1$ and $|\zeta_1 - \zeta_2| \leq 1$. Let $\bar{\zeta}_1 = \frac{|\zeta_2|}{|\zeta_1|}\zeta_1$ and $\zeta \in \mathbb{R}^3$. We break the estimate into three parts:

$$(3.13) \quad |III(x, \zeta) - III(y, \zeta)| \leq C(1 + d_0^{-1})^2 \|f\|_{L_{x,\zeta}^\infty} |x_0 - x_1|^{\frac{1}{2}},$$

$$(3.14) \quad |III(x, \bar{\zeta}_1) - III(x, \zeta_2)| \leq C(1 + d_0^{-1})^3 \|f\|_{L_{x,\zeta}^\infty} |\zeta_1 - \zeta_2|^{\frac{1}{2}},$$

$$(3.15) \quad |III(x, \bar{\zeta}_1) - III(x, \zeta_1)| \leq C(1 + d_0^{-1}) \|f\|_{L_{x,\zeta}^\infty} |\zeta_1 - \zeta_2|^{\frac{1}{2}}.$$

We start with (3.13). We will assume $\tau_-(y, \zeta) \geq \tau_-(x, \zeta)$ and use Proposition 3.1 and the notations in its proof. When $|x - y| < \frac{d_0}{2}$,

$$(3.16) \quad \begin{aligned} & |III(x, \zeta) - III(y, \zeta)| \leq \\ & \left| \int_0^{\frac{|x - X_0|}{|\zeta|}} e^{-\nu s} [G(x - s\zeta, \zeta) - G(y - s\zeta, \zeta)] ds \right| + \left| \int_{\tau_-(x, \zeta)}^{\tau_-(y, \zeta)} e^{-\nu s} G(y - s\zeta, \zeta) ds \right| \\ & \leq C \|f\|_{L_{x,\zeta}^\infty} |x - y|^{\frac{1}{2}} + C(1 + d_0^{-1}) \frac{|x - y|}{|\zeta|} e^{-\nu_0 \frac{d_0}{2|\zeta|}} \|f\|_{L_{x,\zeta}^\infty} \\ & \leq C(1 + d_0^{-1})^2 \|f\|_{L_{x,\zeta}^\infty} |x_0 - x_1|^{\frac{1}{2}}. \end{aligned}$$

Notice that we used the fact $\frac{1}{|\zeta|}e^{-\nu_0 \frac{d_0}{2|\zeta|}} \leq \frac{C}{\nu_0 d_0}$ and $|x - y| \leq 1$ in the last inequality above. When $|x - y| \geq \frac{d_0}{2|\zeta|}$,

$$\begin{aligned}
|III| &\leq C\|f\|_{L_{x,\zeta}^\infty} \\
(3.17) \quad &\leq C\|f\|_{L_{x,\zeta}^\infty} \frac{|x - y|^{\frac{1}{2}}}{d_0^{\frac{1}{2}}} \\
&\leq C(1 + d_0^{-1})^2\|f\|_{L_{x,\zeta}^\infty}|x_0 - x_1|^{\frac{1}{2}}.
\end{aligned}$$

As for (3.14), we may assume $\tau_-(x, \bar{\zeta}_1) \leq \tau_-(x, \zeta_2)$ without loss of generality and adopt the notations in the proof of Proposition 3.3 after replacing ζ_1 by $\bar{\zeta}_1$ in the expression. Then,

$$\begin{aligned}
|III(x, \bar{\zeta}_1) - III(x, \zeta_2)| &\leq \\
(3.18) \quad &\left| \int_0^{\frac{|P_1 x|}{|\zeta_2|}} e^{-\nu(|\zeta_2|)s} [G(x - s\bar{\zeta}_1, \bar{\zeta}_1) - G(x - s\zeta_2, \zeta_2)] ds \right| \\
&+ \left| \int_{\tau_-(x, \bar{\zeta}_1)}^{\tau_-(x, \zeta_2)} e^{-\nu(|\zeta_2|)s} G(x - s\zeta_2, \zeta) ds \right| \\
&=: D_d + R_m.
\end{aligned}$$

When $|x - y| \geq \frac{d_0}{2}$, the desired estimate for R_m holds similar to (3.17). When $|x - y| < \frac{d_0}{2}$, we use Proposition 3.3 and obtain

$$\begin{aligned}
|R_m| &\leq C\|f\|_{L_{x,\zeta}^\infty} (1 + d_0^{-1})\theta|\zeta_2|^{-1}e^{-\nu_0 \frac{d_0}{2|\zeta_2|}} \\
(3.19) \quad &\leq C\|f\|_{L_{x,\zeta}^\infty} (1 + d_0^{-1})|\bar{\zeta}_1 - \zeta_2||\zeta_2|^{-2}e^{-\nu_0 \frac{d_0}{2|\zeta_2|}} \\
&\leq C\|f\|_{L_{x,\zeta}^\infty} (1 + d_0^{-1})^3|\bar{\zeta}_1 - \zeta_2| \\
&\leq C\|f\|_{L_{x,\zeta}^\infty} (1 + d_0^{-1})^3(|\bar{\zeta}_1 - \zeta_1| + |\zeta_1 - \zeta_2|) \\
&\leq C\|f\|_{L_{x,\zeta}^\infty} (1 + d_0^{-1})^3|\zeta_1 - \zeta_2|.
\end{aligned}$$

For D_d , we will subtract and add one term and then apply Lemma 2.1 and Proposition 2.4:

$$\begin{aligned}
(3.20) \quad D_d &\leq \left| \int_0^{\frac{|P_1 x|}{|\zeta_2|}} e^{-\nu(|\zeta_2|)s} (G(x - s\bar{\zeta}_1, \bar{\zeta}_1) - G(x - s\zeta_2, \bar{\zeta}_1)) ds \right| \\
&\quad + \left| \int_0^{\frac{|P_1 x|}{|\zeta_2|}} e^{-\nu(|\zeta_2|)s} (G(x - s\zeta_2, \bar{\zeta}_1) - G(x - s\zeta_2, \zeta_2)) ds \right| \\
&\leq C \|f\|_{L_{x,\zeta}^\infty} \left| \int_0^{\frac{|P_1 x|}{|\zeta_2|}} e^{-\nu(|\zeta_2|)s} ((|\bar{\zeta}_1 - \zeta_2|s)^{\frac{1}{2}} + |\bar{\zeta}_1 - \zeta_2|) ds \right| \\
&\leq C \|f\|_{L_{x,\zeta}^\infty} |\zeta_1 - \zeta_2|^{\frac{1}{2}}.
\end{aligned}$$

Now, we proceed to proving (3.15). we assume $|\bar{\zeta}_1| \leq |\zeta_1|$. We write

$$\begin{aligned}
(3.21) \quad &|III(x, \bar{\zeta}_1) - III(x, \zeta_1)| \leq \\
&\left| \int_0^{\tau_-(x, \zeta_1)} e^{-\nu(|\zeta_2|)s} G(x - s\bar{\zeta}_1, \bar{\zeta}_1) - e^{-\nu(|\zeta_1|)s} G(x - s\zeta_1, \zeta_1) ds \right| \\
&+ \left| \int_{\tau_-(x, \zeta_1)}^{\tau_-(x, \bar{\zeta}_1)} e^{-\nu(|\zeta_2|)s} G(x - s\bar{\zeta}_1, \bar{\zeta}_1) ds \right| \\
&=: D_v + R_v.
\end{aligned}$$

We discuss R_v in two cases. When $|\zeta_1| > 2|\zeta_1 - \zeta_2|^{\frac{1}{2}}$, we notice that $|\zeta_2| > |\zeta_1 - \zeta_2|^{\frac{1}{2}}$. Then,

$$\begin{aligned}
(3.22) \quad R_v &\leq C \|f\|_{L_{x,\zeta}^\infty} \frac{|\zeta_1| - |\zeta_2|}{|\zeta_1||\zeta_2|} e^{-\frac{\nu_0 d_0}{|\zeta_1|}} \\
&\leq C d_0^{-1} \|f\|_{L_{x,\zeta}^\infty} \frac{|\zeta_1| - |\zeta_2|}{|\zeta_2|} \\
&\leq C d_0^{-1} \|f\|_{L_{x,\zeta}^\infty} |\zeta_1 - \zeta_2|^{\frac{1}{2}}.
\end{aligned}$$

When $|\zeta_1| \leq 2|\zeta_1 - \zeta_2|^{\frac{1}{2}}$, $|\zeta_2| \leq 2|\zeta_1 - \zeta_2|^{\frac{1}{2}}$ because we assume $|\zeta_2| \leq |\zeta_1|$. Therefore,

$$\begin{aligned}
(3.23) \quad R_v &\leq C \|f\|_{L_{x,\zeta}^\infty} \int_{\tau_-(x, \zeta_1)}^{\tau_-(x, \bar{\zeta}_1)} e^{-\nu_0 s} ds \\
&\leq C \|f\|_{L_{x,\zeta}^\infty} \left(e^{-\frac{\nu_0 d_0}{|\zeta_1|}} + e^{-\frac{\nu_0 d_0}{|\zeta_2|}} \right) \\
&\leq C \|f\|_{L_{x,\zeta}^\infty} \left(\frac{|\zeta_1|}{\nu_0 d_0} + \frac{|\zeta_2|}{\nu_0 d_0} \right) \\
&\leq C d_0^{-1} \|f\|_{L_{x,\zeta}^\infty} |\zeta_1 - \zeta_2|^{\frac{1}{2}}.
\end{aligned}$$

As for D_v ,

$$\begin{aligned}
(3.24) \quad |D_v| &\leq \int_0^{\frac{|xP_1|}{|\zeta_1|}} \left| e^{-\nu(|\zeta_2|)s} - e^{-\nu(|\zeta_1|)s} \right| |G(x - s\bar{\zeta}_1, \bar{\zeta}_1)| ds \\
&\quad + \int_0^{\frac{|xP_1|}{|\zeta_1|}} e^{-\nu(|\zeta_1|)s} |G(x - s\bar{\zeta}_1, \bar{\zeta}_1) - G(x - s\zeta_1, \bar{\zeta}_1)| ds \\
&\quad + \int_0^{\frac{|xP_1|}{|\zeta_1|}} e^{-\nu(|\zeta_1|)s} |G(x - s\zeta_1, \bar{\zeta}_1) - G(x - s\zeta_1, \zeta_1)| ds \\
&=: D_v^1 + D_v^2 + D_v^3.
\end{aligned}$$

We can deal with D_v^2 and D_v^3 similar to (3.20). To deal with D_v^1 , we need to discuss the derivative of ν :

$$\begin{aligned}
(3.25) \quad |\nabla \nu_\gamma(|\zeta|)| &= \left| \beta_0 \int_{\mathbb{R}^3} e^{-|\eta|^2} |\eta - \zeta|^{\gamma-1} \frac{\eta - \zeta}{|\eta - \zeta|} d\eta \right| \\
&\leq C(1 + |\zeta|)^{\gamma-1}.
\end{aligned}$$

Applying the Mean Value Theorem, there exists z between $|\zeta_1|$ and $|\zeta_2|$ such that

$$\begin{aligned}
(3.26) \quad D_v^1 &\leq C \int_0^{\frac{|xP_1|}{|\zeta_1|}} ||\zeta_2| - |\zeta_1|| e^{-\nu(z)s} (1+z)^{\gamma-1} |G(x - s\bar{\zeta}_1, \bar{\zeta}_1)| ds \\
&\leq C \|f\|_{L_{x,\zeta}^\infty} |\zeta_1 - \zeta_2|.
\end{aligned}$$

Therefore, we prove the regularity of the *III*. \square

4. PRESERVATION OF THE REGULARITY FROM THE BOUNDARY

In this section, we will prove that the damped transport equation preserves the regularity of the boundary data and so does *II*.

Proposition 4.1. *Let $0 < \sigma < \frac{1}{2}$ and Ω be a bounded C^1 convex domain. Suppose $f \in L_{x,\zeta}^\infty$ satisfies*

$$(4.1) \quad |f(X, \zeta) - f(Y, \zeta')| \leq M(|X - Y|^2 + |\zeta - \zeta'|^2)^{\frac{\sigma}{2}}$$

for any $(X, \zeta), (Y, \zeta') \in \Gamma_-$. Then, there exists a constant C_9 depending only on $\|f\|_{L_{x,\zeta}^\infty}$, M , Ω , and the potential such that, I from (1.21) satisfies

$$(4.2) \quad |I(x, \zeta) - I(y, \zeta)| \leq C_9(1 + d_0^{-1})^{2\sigma} |x - y|^\sigma,$$

$$(4.3) \quad |I(x, \zeta_1) - I(x, \zeta_2)| \leq C_9(1 + d_0^{-1})^2 |\zeta_1 - \zeta_2|^\sigma.$$

As a consequence,

$$(4.4) \quad |I(x, \zeta_1) - I(y, \zeta_2)| \leq 2C_9(1 + d_0^{-1})^2 (|x - y|^2 + |\zeta_1 - \zeta_2|^2)^{\frac{\sigma}{2}}.$$

Proof. We can see that I is bounded and therefore the proposition holds if $|x - y| > 1$ or $|\zeta_1 - \zeta_2| > 1$. We only need to discuss when $|x - y| \leq 1$ and $|\zeta_1 - \zeta_2| \leq 1$.

We start with (4.2). Notice that here we are picky about the growth rate with respect to d_0 because that it is needed for the proof of regularity of II . Notice that if $d_0 < |x - y|$, then $1 < d_0^{-1}|x - y|$ and therefore the inequality holds. Hence, we only need to discuss the case $|x - y| < d_0$.

(4.5)

$$\begin{aligned} |I(x, \zeta) - I(y, \zeta)| &= \left| f(p(x, \zeta), \zeta) e^{-\nu(|\zeta|)\tau_-(x, \zeta)} - f(p(y, \zeta), \zeta) e^{-\nu(|\zeta|)\tau_-(y, \zeta)} \right| \\ &\leq |f(p(x, \zeta), \zeta) - f(p(y, \zeta), \zeta)| e^{-\nu(|\zeta|)\tau_-(x, \zeta)} \\ &\quad + |f(p(y, \zeta), \zeta)| \left| e^{-\nu(|\zeta|)\tau_-(x, \zeta)} - e^{-\nu(|\zeta|)\tau_-(y, \zeta)} \right| \\ &=: I_a + I_b. \end{aligned}$$

From (3.1) and our assumption, we have

$$(4.6) \quad |I_a| \leq CM(1 + d_0^{-1})^\sigma |x - y|^\sigma.$$

To deal with I_b , we need to discuss the following two cases. If $|\zeta| \geq (1 + d_0^{-1})^{1-2\sigma} |x - y|^{1-\sigma}$, then

(4.7)

$$\begin{aligned} |e^{-\nu(|\zeta|)\tau_-(x, \zeta)} - e^{-\nu(|\zeta|)\tau_-(y, \zeta)}| &\leq C(1 + |\zeta|)^\gamma |\tau_-(x, \zeta) - \tau_-(y, \zeta)| \\ &\leq C((1 + d_0^{-1})(1 + |\zeta|)^\gamma \frac{|x - y|}{|\zeta|}) \leq C \begin{cases} (1 + d_0^{-1})|x - y|, & |\zeta| > 1 \\ (1 + d_0^{-1})^{2\sigma} |x - y|^\sigma, & |\zeta| \leq 1 \end{cases} \\ &\leq C(1 + d_0^{-1})^{2\sigma} |x - y|^\sigma. \end{aligned}$$

Notice that $0 \leq \gamma < 1$ and $|x - y| < d_0$ were used in the above inequalities. For the case $|\zeta| < (1 + d_0^{-1})^{1-2\sigma} |x - y|^{1-\sigma}$,

$$\begin{aligned} |e^{-\nu\tau_-(x, \zeta)} - e^{-\nu\tau_-(y, \zeta)}| &\leq 2e^{-\nu_0 \frac{d_0}{|\zeta|}} \\ (4.8) \quad &\leq 2 \left(\frac{|\zeta|}{\nu_0 d_0} \right)^{\frac{\sigma}{1-\sigma}} \left(\frac{\nu_0 d_0}{|\zeta|} \right)^{\frac{\sigma}{1-\sigma}} e^{-\nu_0 \frac{d_0}{|\zeta|}} \\ &\leq C \left(\frac{|\zeta|}{\nu_0 d_0} \right)^{\frac{\sigma}{1-\sigma}} \leq C(1 + d_0^{-1})^{2\sigma} |x - y|^\sigma. \end{aligned}$$

For (4.3), we adopt the notation in the proof of Proposition 3.3. We introduce $\bar{\zeta}_1 = \frac{|\zeta_2|}{|\zeta_1|} \zeta_1$ and write

$$(4.9) \quad \begin{aligned} |I(x, \zeta_1) - I(x, \zeta_2)| &\leq |I(x, \zeta_1) - I(x, \bar{\zeta}_1)| + |I(x, \bar{\zeta}_1) - I(x, \zeta_2)| \\ &=: \Delta I_r + \Delta I_\theta. \end{aligned}$$

For the former part,

$$\begin{aligned}
\Delta I_r &= \left| f(P_1, \zeta_1) e^{-\nu(|\zeta_1|) \frac{\overline{xP_1}}{|\zeta_1|}} - f(P_1, \bar{\zeta}_1) e^{-\nu(|\zeta_2|) \frac{\overline{xP_1}}{|\zeta_2|}} \right| \\
(4.10) \quad &\leq \left| [f(P_1, \zeta_1) - f(P_1, \bar{\zeta}_1)] e^{-\nu(|\zeta_1|) \frac{\overline{xP_1}}{|\zeta_1|}} \right| \\
&\quad + \left| f(P_1, \bar{\zeta}_1) \left[e^{-\nu(|\zeta_1|) \frac{\overline{xP_1}}{|\zeta_1|}} - e^{-\nu(|\zeta_2|) \frac{\overline{xP_1}}{|\zeta_2|}} \right] \right| \\
&=: \Delta I_r^1 + \Delta I_r^2.
\end{aligned}$$

We can obtain the desired bound for ΔI_r^1 directly from the assumption. For ΔI_r^2 , we need to discuss in two cases. First, we consider the case $|\zeta_1| \leq 2||\zeta_1| - |\zeta_2||^{\frac{1-\sigma}{2}}$. Notice that $|\zeta_2| \leq 3||\zeta_1| - |\zeta_2||^{\frac{1-\sigma}{2}}$. Similar to (4.8), we have

$$\begin{aligned}
(4.11) \quad &|e^{-\nu(|\zeta_1|) \frac{\overline{xP_1}}{|\zeta_1|}}| + |e^{-\nu(|\zeta_2|) \frac{\overline{xP_1}}{|\zeta_2|}}| \leq e^{-\frac{\nu_0 d_0}{|\zeta_1|}} + e^{-\frac{\nu_0 d_0}{|\zeta_2|}} \\
&\leq C \left(\left| \frac{|\zeta_1|}{\nu_0 d_0} \right|^{\frac{2\sigma}{1-\sigma}} + \left| \frac{|\zeta_2|}{\nu_0 d_0} \right|^{\frac{2\sigma}{1-\sigma}} \right) \\
&\leq C \left(\frac{1}{\nu_0 d_0} \right)^{\frac{2\sigma}{1-\sigma}} ||\zeta_1| - |\zeta_2||^\sigma \\
&\leq C(1 + d_0^{-1})^2 |\zeta_1 - \zeta_2|^\sigma.
\end{aligned}$$

When $|\zeta_1| > 2||\zeta_1| - |\zeta_2||^{\frac{1-\sigma}{2}}$, we have $|\zeta_2| > ||\zeta_1| - |\zeta_2||^{\frac{1-\sigma}{2}}$. By the Mean Value Theorem, there exists η' between $|\zeta_1|$ and $|\zeta_2|$, therefore $\eta' > ||\zeta_1| - |\zeta_2||^{\frac{1-\sigma}{2}}$, such that,

$$\begin{aligned}
(4.12) \quad &|\Delta I_r^2| = \left| f(P_1, \bar{\zeta}_1) \overline{xP_1} (|\zeta_1| - |\zeta_2|) e^{-\nu(\eta') \frac{\overline{xP_1}}{\eta'}} \frac{d}{d\eta} \left(\frac{\nu(\eta)}{\eta} \right) \right|_{\eta=\eta'} \\
&\leq ||\zeta_1| - |\zeta_2|| \begin{cases} \frac{1}{\eta'^2}, & 0 < \eta' < 1, \\ \frac{1}{\eta'^{2-\gamma}}, & \eta' \geq 1 \end{cases} \\
&\leq C ||\zeta_1| - |\zeta_2||^\sigma \leq C |\zeta_1 - \zeta_2|^\sigma.
\end{aligned}$$

Notice that we used the estimate (3.25) and the fact $|\zeta_1 - \zeta_2| \leq 1$.

Now, we proceed to ΔI_θ .

$$\begin{aligned}
(4.13) \quad &|\Delta I_\theta| = \left| f(P_1, \bar{\zeta}_1) e^{-\nu(|\zeta_2|) \frac{\overline{xP_1}}{|\zeta_2|}} - f(P_2, \zeta_2) e^{-\nu(|\zeta_2|) \frac{\overline{xP_2}}{|\zeta_2|}} \right| \\
&\leq |f(P_1, \bar{\zeta}_1) - f(P_2, \zeta_2)| e^{-\nu(|\zeta_2|) \frac{\overline{xP_1}}{|\zeta_2|}} \\
&\quad + |f(P_2, \zeta_2)| \left| e^{-\nu(|\zeta_2|) \frac{\overline{xP_1}}{|\zeta_2|}} - e^{-\nu(|\zeta_2|) \frac{\overline{xP_2}}{|\zeta_2|}} \right| \\
&=: \Delta I_\theta^1 + \Delta I_\theta^2.
\end{aligned}$$

Applying Proposition 3.3 and our assumption, we have

$$\begin{aligned}
|\Delta I_\theta^1| &\leq C[(1 + d_0^{-1})^\sigma |\theta|^\sigma e^{-\nu_0 \frac{d_0}{|\zeta_2|}} + |\bar{\zeta}_1 - \zeta_2|^\sigma] \\
(4.14) \quad &\leq C[(1 + d_0^{-1})^\sigma \left| \frac{|\bar{\zeta}_1 - \zeta_2|}{|\zeta_2|} \right|^\sigma e^{-\nu_0 \frac{d_0}{|\zeta_2|}} + |\bar{\zeta}_1 - \zeta_2|^\sigma] \\
&\leq C(1 + d_0^{-1})^{2\sigma} |\zeta_1 - \zeta_2|^\sigma.
\end{aligned}$$

For ΔI_θ^2 , we again break it into two cases: $|\zeta_2| \leq |\bar{\zeta}_1 - \zeta_2|^{\frac{1-\sigma}{2}}$ and $|\zeta_2| > |\bar{\zeta}_1 - \zeta_2|^{\frac{1-\sigma}{2}}$. We mimic (4.11) and obtain the desired bound for the former case. For the latter one, applying the Mean Value Theorem and Proposition 3.3, we have

$$\begin{aligned}
(4.15) \quad &|e^{-\nu(|\zeta_2|) \frac{|xP_1|}{|\zeta_2|}} - e^{-\nu(|\zeta_2|) \frac{|xP_2|}{|\zeta_2|}}| \leq \frac{\nu(|\zeta_2|)}{|\zeta_2|} |\overline{xP_1} - \overline{xP_2}| \\
&\leq C(1 + d_0^{-1}) \frac{\nu(|\zeta_2|)}{|\zeta_2|} \frac{|\bar{\zeta}_1 - \zeta_2|}{|\zeta_2|} \leq C(1 + d_0^{-1}) \begin{cases} |\bar{\zeta}_1 - \zeta_2|^\sigma, & |\zeta_2| < 1 \\ |\zeta_1 - \zeta_2|, & |\zeta_2| \geq 1 \end{cases} \\
&\leq C(1 + d_0^{-1}) |\zeta_1 - \zeta_2|^\sigma.
\end{aligned}$$

Notice that the last inequality above holds because we are now discussing the case when $|\zeta_1 - \zeta_2| \leq 1$. Hence, we complete the proof. \square

Now we only have II left.

Proposition 4.2. *With the same assumption as Proposition 4.1 above, there exists a constant C_{10} only depending on $\|f\|_{L_{x,\zeta}^\infty}$, M , σ , Ω , and the potential such that,*

$$(4.16) \quad |II(x, \zeta_1) - II(y, \zeta_2)| \leq C_{10}(1 + d_0^{-1})^3(|x - y|^2 + |\zeta_1 - \zeta_2|^2)^{\frac{\sigma}{2}}.$$

Proof. Notice that the inequality holds when $|x - y|$ or $|\zeta_1 - \zeta_2|$ are large compare with d_0 . We only need the deal the case when both of them are small. We can break the proof into the following three estimates:

$$(4.17) \quad |II(x, \zeta) - II(y, \zeta)| \leq C(1 + d_0^{-1})^2 |x - y|^\sigma,$$

$$(4.18) \quad |II(x, \bar{\zeta}_1) - II(x, \zeta_2)| \leq C(1 + d_0^{-1})^3 |\zeta_1 - \zeta_2|^\sigma,$$

$$(4.19) \quad |II(x, \bar{\zeta}_1) - II(x, \zeta_1)| \leq C(1 + d_0^{-1})^2 |\zeta_1 - \zeta_2|^\sigma.$$

In the proof of (4.17), we may assume $\tau_-(y, \zeta) \geq \tau_-(x, \zeta)$. We will apply estimate (4.2) in Proposition 4.1 in our proof. Notice that (4.2) depends on the distance and breaks down at boundary. However, thank to the help of the integration over s in II , we still can have the desired estimate. To carry out the analysis, we first dyadically decompose Ω as follows. Let $M_n = \{x \in \Omega | d(x, \partial\Omega) \leq d_0 2^{-n}\}$ for integer $n \geq 1$. Let $D_0 = \Omega \setminus M_1$ and $D_n = M_n \setminus M_{n+1}$ for $n \geq 1$. Let N be the integer such that $8|x - y| > 2^{-N}d_0 \geq 4|x - y|$. D_n for $0 \leq n \leq N$ together with M_{N+1} form a decomposition of Ω . Let

$X_0 = p(x, \zeta)$ and $Y_0 = p(y, \zeta)$. Let Z_n be the intersection of $\overline{xX_0}$ with $\partial M_n \setminus \partial\Omega$ for $n \geq 1$ and $Z_0 = x$. Applying Proposition 3.2, we have

$$(4.20) \quad |\overline{XZ_n}| \leq R d_0^{-1} 2^{-n} d_0 \leq R 2^{-n}.$$

For $n \geq 1$,

$$(4.21) \quad \begin{aligned} \Delta II_d^n &:= \left| \int_{\frac{\overline{Z_n x}}{|\zeta|}}^{\frac{\overline{Z_{n+1} x}}{|\zeta|}} e^{-\nu s} \int_{\mathbb{R}^3} k(\zeta, \zeta') [I(x - s\zeta, \zeta') - I(y - s\zeta, \zeta')] d\zeta' ds \right| \\ &\leq C \left| \int_{\frac{\overline{Z_n x}}{|\zeta|}}^{\frac{\overline{Z_{n+1} x}}{|\zeta|}} e^{-\nu s} \int_{\mathbb{R}^3} k(\zeta, \zeta') (1 + 2^{n+2} d_0^{-1})^{2\sigma} |x - y|^\sigma d\zeta' ds \right| \\ &\leq C (1 + 2^{n+2} d_0^{-1})^{2\sigma} |x - y|^\sigma \left| \int_{\frac{\overline{Z_n x}}{|\zeta|}}^{\frac{\overline{Z_{n+1} x}}{|\zeta|}} e^{-\nu s} ds \right| \\ &\leq C (1 + 2^{n+2} d_0^{-1})^{2\sigma} |x - y|^\sigma R 2^{-n} |\zeta|^{-1} e^{-\frac{\nu_0 d_0}{2|\zeta|}} \\ &\leq C 2^{-(1-2\sigma)n} (1 + d_0^{-1})^{1+2\sigma} |x - y|^\sigma. \end{aligned}$$

Notice that $1 - 2\sigma > 0$ and therefore

$$(4.22) \quad \sum_{n=1}^N II_d^n \leq C (1 + d_0^{-1})^{1+2\sigma} |x - y|^\sigma.$$

For the segment within D_0 ,

$$(4.23) \quad \begin{aligned} \Delta II_d^0 &:= \left| \int_0^{\frac{\overline{Z_1 x}}{|\zeta|}} e^{-\nu(|\zeta|)s} \int_{\mathbb{R}^3} k(\zeta, \zeta') [I(x - s\zeta, \zeta') - I(y - s\zeta, \zeta')] d\zeta' ds \right| \\ &\leq C (1 + 4d_0^{-1})^{2\sigma} |x - y|^\sigma \int_0^{\frac{\overline{Z_1 x}}{|\zeta|}} e^{-\nu(|\zeta|)s} \int_{\mathbb{R}^3} |k(\zeta, \zeta')| d\zeta' ds \\ &\leq C (1 + d_0^{-1})^{2\sigma} |x - y|^\sigma \left(\frac{1}{\nu(|\zeta|)} - \frac{1}{\nu(|\zeta|)} e^{-\nu(|\zeta|) \frac{\overline{Z_1 x}}{|\zeta|}} \right) \\ &\leq C (1 + d_0^{-1})^{2\sigma} |x - y|^\sigma. \end{aligned}$$

For the remaining part, we have

$$\begin{aligned}
(4.24) \quad & \left| \int_{\frac{|\overline{z_{N+1}x}|}{|\zeta|}}^{\tau_-(x,\zeta)} e^{-\nu s} \int_{\mathbb{R}^3} k(\zeta, \zeta') I(x - s\zeta, \zeta') d\zeta' ds \right| \\
& + \left| \int_{\frac{|\overline{z_{N+1}x}|}{|\zeta|}}^{\tau_-(y,\zeta)} e^{-\nu s} \int_{\mathbb{R}^3} k(\zeta, \zeta') I(y - s\zeta, \zeta') d\zeta' ds \right| \\
& \leq C(1 + d_0^{-1}) \frac{|x - y|}{|\zeta|} e^{-\frac{\nu_0 d_0}{2|\zeta|}} \\
& \leq C(1 + d_0^{-1})^2 |x - y|,
\end{aligned}$$

where the definition of N and Proposition 3.1 were used. Hence, we proved (4.17).

Now, we proceed to (4.18). Let θ be the angle between $\bar{\zeta}_1$ and ζ_2 . Without lost of generality, we may assume $\tau_-(x, \bar{\zeta}_1) \leq \tau_-(x, \zeta_2)$. Let $P_1 = p(x, \bar{\zeta}_1)$ and $P_2 = p(x, \zeta_2)$. Again, we only need to prove the case when $|\bar{\zeta}_1 - \zeta_2| \leq 1$. Let θ be the angle between ζ_1 and ζ_2 . We adopt the same dyadic decomposition above and stop at N' such that $8|\overline{P_1x}|\theta > 2^{-N'}d_0 \geq 4|\overline{P_1x}|\theta$. Let S_n be the intersection between $\overline{P_1x}$ and $\partial M_n \setminus \partial\Omega$.

$$\begin{aligned}
(4.25) \quad \Delta II^n &:= \left| \int_{\frac{|\overline{xS_n}|}{|\zeta_2|}}^{\frac{|\overline{xS_{n+1}}|}{|\zeta_2|}} e^{-\nu s} \int_{\mathbb{R}^3} k(\bar{\zeta}_1, \zeta') I(x - s\bar{\zeta}_1, \zeta') - k(\zeta_2, \zeta') I(x - s\zeta_2, \zeta') d\zeta' ds \right| \\
&\leq \left| \int_{\frac{|\overline{xS_n}|}{|\zeta_2|}}^{\frac{|\overline{xS_{n+1}}|}{|\zeta_2|}} e^{-\nu s} \int_{\mathbb{R}^3} k(\bar{\zeta}_1, \zeta') [I(x - s\bar{\zeta}_1, \zeta') - I(x - s\zeta_2, \zeta')] d\zeta' ds \right| \\
&\quad + \left| \int_{\frac{|\overline{xS_n}|}{|\zeta_2|}}^{\frac{|\overline{xS_{n+1}}|}{|\zeta_2|}} e^{-\nu s} \int_{\mathbb{R}^3} [k(\bar{\zeta}_1, \zeta') - k(\zeta_2, \zeta')] I(x - s\zeta_2, \zeta') d\zeta' ds \right| \\
&=: DI^n + DK^n.
\end{aligned}$$

DI^n can be bounded by a similar argument:

(4.26)

$$\begin{aligned}
DI^n &\leq C \left| \int_{\frac{|xS_n|}{|\zeta_2|}}^{\frac{|xS_{n+1}|}{|\zeta_2|}} e^{-\nu s} \int_{\mathbb{R}^3} k(\bar{\zeta}_1, \zeta') (1 + 2^{n+2} d_0^{-1})^{2\sigma} \left(\frac{|\bar{\zeta}_1 - \zeta_2|}{|\zeta_2|} \right)^\sigma d\zeta' ds \right| \\
&\leq C (1 + 2^{n+2} d_0^{-1})^{2\sigma} \left(\frac{|\bar{\zeta}_1 - \zeta_2|}{|\zeta_2|} \right)^\sigma \left| \int_{\frac{|xS_n|}{|\zeta_2|}}^{\frac{|xS_{n+1}|}{|\zeta_2|}} e^{-\nu s} ds \right| \\
&\leq C (1 + 2^{n+2} d_0^{-1})^{2\sigma} \left(\frac{|\bar{\zeta}_1 - \zeta_2|}{|\zeta_2|} \right)^\sigma R 2^{-n} |\zeta_2|^{-1} e^{-\frac{\nu_0 d_0}{2|\zeta_2|}} \\
&\leq C 2^{-(1-2\sigma)n} (1 + d_0^{-1})^{1+3\sigma} |\bar{\zeta}_1 - \zeta_2|^\sigma.
\end{aligned}$$

Therefore,

$$(4.27) \quad \sum_1^{N'} DI^n \leq C (1 + d_0^{-1})^{\frac{5}{2}} |\bar{\zeta}_1 - \zeta_2|^\sigma.$$

For DK^n part, from the assumption, we observe that $I(x, \zeta)$ is bounded. Therefore, we have

$$(4.28) \quad \left| \int_{\mathbb{R}^3} [k(\bar{\zeta}_1, \zeta') - k(\zeta_2, \zeta')] I(x - s\zeta_2, \zeta') d\zeta' \right| \leq C |\bar{\zeta}_1 - \zeta_2|.$$

Hence,

$$\begin{aligned}
DK^n &\leq C |\bar{\zeta}_1 - \zeta_2| \left| \int_{\frac{|xS_n|}{|\zeta_2|}}^{\frac{|xS_{n+1}|}{|\zeta_2|}} e^{-\nu s} ds \right| \\
(4.29) \quad &\leq C |\bar{\zeta}_1 - \zeta_2| \frac{R 2^{-n}}{|\zeta_2|} e^{-\frac{\nu_0 d_0}{2|\zeta_2|}} \\
&\leq C (1 + d_0^{-1}) 2^{-n} |\bar{\zeta}_1 - \zeta_2|.
\end{aligned}$$

As a result,

$$(4.30) \quad \sum_{n=1}^{N'} DK_n \leq C (1 + d_0^{-1}) |\bar{\zeta}_1 - \zeta_2|.$$

For the segment within D_0 ,

(4.31)

$$\begin{aligned}
\Delta II^0 &:= \left| \int_0^{\frac{|xS_1|}{|\zeta_2|}} e^{-\nu s} \int_{\mathbb{R}^3} k(\bar{\zeta}_1, \zeta') I(x - s\bar{\zeta}_1, \zeta') - k(\zeta_2, \zeta') I(x - s\zeta_2, \zeta') d\zeta' ds \right| \\
&\leq C \int_0^{\frac{|xS_1|}{|\zeta_2|}} e^{-\nu s} [(1 + d_0^{-1})^{2\sigma} (s|\bar{\zeta}_1 - \zeta_2|)^\sigma + |\bar{\zeta}_1 - \zeta_2|] ds \\
&\leq C (1 + d_0^{-1})^{2\sigma} |\bar{\zeta}_1 - \zeta_2|^\sigma.
\end{aligned}$$

Similarly, we can also establish bounds for the remaining part because the domain of integration is small and distance from x is big. Hence, we proved (4.18).

Finally we are going to prove (4.19). We also need the dyadic decomposition used above. Let \bar{N} be the integer such that

$$(4.32) \quad 4|\zeta_2 - \zeta_1| \geq 2^{-\bar{N}} d_0 \geq 2|\zeta_2 - \zeta_1|.$$

Without lost generality, we may assume $|\bar{\zeta}_1| \leq |\zeta_1|$. Let $S_0 = x$. For $1 \leq n \leq \bar{N}$,

$$(4.33) \quad \begin{aligned} DV^n &:= \left| \int_{\frac{|S_n x|}{|\zeta_1|}}^{\frac{|S_{n+1} x|}{|\zeta_1|}} \int_{\mathbb{R}^3} e^{-\nu(|\zeta_2|)s} k(\bar{\zeta}_1, \zeta') I(x - s\bar{\zeta}_1, \zeta') - e^{-\nu(|\zeta_1|)s} k(\zeta_1, \zeta') I(x - s\zeta_1, \zeta') d\zeta' ds \right| \\ &\leq \left| \int_{\frac{|S_n x|}{|\zeta_1|}}^{\frac{|S_{n+1} x|}{|\zeta_1|}} \int_{\mathbb{R}^3} \left[e^{-\nu(|\zeta_2|)s} - e^{-\nu(|\zeta_1|)s} \right] k(\bar{\zeta}_1, \zeta') I(x - s\bar{\zeta}_1, \zeta') \right. \\ &\quad \left. + e^{-\nu(|\zeta_1|)s} [k(\bar{\zeta}_1, \zeta') - k(\zeta_1, \zeta')] I(x - s\bar{\zeta}_1, \zeta') \right. \\ &\quad \left. + e^{-\nu(|\zeta_1|)s} k(\zeta_1, \zeta') [I(x - s\bar{\zeta}_1, \zeta') - I(x - s\zeta_1, \zeta')] d\zeta' ds \right| \\ &\leq C \left[(1 + d_0^{-1}) |\zeta_2 - \zeta_1| 2^{-n} + (1 + d_0^{-1})^{1+2\sigma} |\zeta_2 - \zeta_1|^\sigma 2^{-(1-2\sigma)n} \right]. \end{aligned}$$

Hence,

$$(4.34) \quad \sum_{n=1}^{\bar{N}} DV^n \leq C(1 + d_0^{-1})^{1+2\sigma} |\zeta_2 - \zeta_1|^\sigma.$$

We can bound the contribution from the segment within D_0 similar to (4.23) and (4.31).

For the remaining part, we define

$$(4.35) \quad \begin{aligned} RV &:= \left| \int_{\frac{|S_{\bar{N}+1} x|}{|\zeta_1|}}^{\frac{|P_1 x|}{|\zeta_2|}} \int_{\mathbb{R}^3} e^{-\nu(|\zeta_2|)s} k(\bar{\zeta}_1, \zeta') I(x - s\bar{\zeta}_1, \zeta') - e^{-\nu(|\zeta_1|)s} k(\zeta_2, \zeta') I(x - s\zeta_2, \zeta') d\zeta' ds \right|. \end{aligned}$$

When $|\zeta_1| > 2\sqrt{|\zeta_1 - \zeta_2|}$, $|\zeta_2| > \sqrt{|\zeta_1 - \zeta_2|}$. Therefore,

(4.36)

$$\begin{aligned}
RV &\leq C \int_{\frac{|\overline{P_1 x}|}{|\zeta_1|}}^{\frac{|\overline{P_1 x}|}{|\zeta_2|}} e^{-\nu_0 s} ds \leq C e^{-\frac{\nu_0 d_0}{2|\zeta_1|}} \left[\frac{|\overline{P_1 S_{\bar{N}+1}}|}{|\zeta_2|} + |\overline{S_{\bar{N}+1} x}| \left(\frac{1}{|\zeta_2|} - \frac{1}{|\zeta_1|} \right) \right] \\
&\leq C e^{-\frac{\nu_0 d_0}{2|\zeta_1|}} \left[(1 + d_0^{-1}) \frac{|\zeta_1 - \zeta_2|}{|\zeta_2|} + \frac{||\zeta_1| - |\zeta_2||}{|\zeta_2||\zeta_1|} \right] \\
&\leq C(1 + d_0^{-1}) \frac{|\zeta_1 - \zeta_2|}{|\zeta_2|} \leq C(1 + d_0^{-1}) |\zeta_1 - \zeta_2|^{\frac{1}{2}}.
\end{aligned}$$

For the other case, $2\sqrt{|\zeta_1 - \zeta_2|} \geq |\zeta_1| \geq |\zeta_2|$.

$$\begin{aligned}
(4.37) \quad RV &\leq C \int_{\frac{|\overline{P_1 x}|}{|\zeta_1|}}^{\frac{|\overline{P_1 x}|}{|\zeta_2|}} e^{-\nu_0 s} ds \leq C(e^{-\nu_0 \frac{|\overline{S_{\bar{N}+1} x}|}{|\zeta_1|}} + e^{-\nu_0 \frac{|\overline{P_1 x}|}{|\zeta_2|}}) \\
&\leq C d_0^{-1} (|\zeta_1| + |\zeta_2|) \leq C d_0^{-1} \sqrt{|\zeta_1 - \zeta_2|}.
\end{aligned}$$

Hence, we finished the proof. \square

5. GAINING OF INTEGRABILITY

We devote this final section to the proof of the following proposition.

Proposition 5.1. *Suppose $f \in L_{x,\zeta}^*$ solves the stationary linearized Boltzmann equation (1.1) for gases with cutoff hard potential or cutoff Maxwellian gases, (1.2), and there exists $\phi(\zeta) \in L_\zeta^*$ such that*

$$(5.1) \quad |f(X, \eta)| < \phi(\zeta),$$

for any $(X, \eta) \in \Gamma_-$. Then, $f \in L_{x,\zeta}^\infty L_\zeta^*$.

Moreover, if we further assume that for a fix $0 < \alpha < \frac{1}{2}$ there exist M such that

$$(5.2) \quad |f(X, \eta) - f(Y, \omega)| \leq M (|\eta - \omega|^2 + |X - Y|^2)^{\frac{\alpha}{2}}$$

for any $(X, \eta), (Y, \omega) \in \Gamma_-$, then $f \in L_{x,\zeta}^\infty$.

To prove $f \in L_x^\infty L_\zeta^*$, we first use the integral equation (1.12) and name the former term on the right hand side as $\tilde{g}(x, \zeta)$ and the latter as $\tilde{f}(x, \zeta)$. From the assumption, we know that both $\|\tilde{g}\|_{L_x^1 L_\zeta^*}$ and $\|\tilde{g}\|_{L_x^\infty L_\zeta^*}$ are bounded, therefore, through interpolation, so is $\|\tilde{g}\|_{L_x^p L_\zeta^*}$ for any $p \in [1, \infty]$. We perform zero extension for $f(x, \zeta)$ outside Ω and, with abuse of notation, still call it f . We use $*$ to denote the convolution in space. The following estimate is to be proved.

Lemma 5.2. *Let $0 < \alpha < 1$. f and \tilde{f} are as defined above. Then, there exists a constant C_{11} only depending on the potential such that*

$$(5.3) \quad \|\tilde{f}(x, \cdot)\|_{L_\zeta^*}^2 \leq C_{11} \left(\frac{\chi_{(0,R)}(|x|)}{|x|^{2-\alpha}} \right) * \|f(x, \cdot)\|_{L_\zeta^*}^2,$$

where R is the diameter of Ω and $\chi_{(0,R)}$ is the characteristic function of $(0, R)$.

Let $1 < p < \frac{3}{2-\alpha}$, $q, r \in [0, \infty]$. Applying Young's inequality to the convolution above, we can conclude that, if

$$(5.4) \quad \frac{2}{r} + 1 = \frac{1}{p} + \frac{2}{q},$$

then $f \in L_x^q L_\zeta^*$ implies $\tilde{f} \in L_x^r L_\zeta^*$. Starting from $q = 2$, through finite times of iteration, we can obtain that $f \in L_x^\infty L_\zeta^*$. Notice that the hard sphere case is barely missed.

The fact $f \in L_x^\infty L_\zeta^*$ together with Hölder continuity assumption on the boundary data implies

$$(5.5) \quad |f(X, \zeta)| \leq C_{12} M^{\frac{3}{3+2\sigma}} \|f\|_{L_x^\infty L_\zeta^*}^{\frac{2\sigma}{3+2\sigma}},$$

where $C_{12} = 2(\frac{3}{4\pi\nu_0})^{\frac{\sigma}{3+2\sigma}}$ for any $(X, \zeta) \in \Gamma_-$. Then, from the integral equation (1.12), we can conclude $f \in L_{x,\zeta}^\infty$.

In order to prove Lemma 5.2, we shall first estimate the decay of $K(f)$:

Proposition 5.3. *For any $0 \leq \gamma \leq 1$,*

$$(5.6) \quad |K(f)| \leq C \|f\|_{L_\zeta^*} (1 + |\zeta|)^{-\frac{3-\gamma}{2}}.$$

The constant C above may depend on the potential.

Proof.

$$(5.7) \quad \begin{aligned} |K(f)(\zeta)| &= \left| \int_{\mathbb{R}^3} k(\zeta, \zeta_*) f(\zeta_*) d\zeta_* \right| \\ &= \left(\int_{\mathbb{R}^3} |k(\zeta, \zeta_*)|^2 \frac{1}{|\nu(\zeta_*)|} d\zeta_* \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nu(\zeta_*)| |f(\zeta_*)|^2 d\zeta_* \right)^{\frac{1}{2}} \\ &\leq \|f\|_{L_\zeta^*} \left(\int_{\mathbb{R}^3} \frac{e^{-\frac{1}{4}(|\zeta - \zeta_*|^2 + (\frac{|\zeta|^2 - |\zeta_*|^2}{|\zeta - \zeta_*|})^2)}}{|\zeta - \zeta_*|^2 (1 + |\zeta| + |\zeta_*|)^{2(1-\gamma)} (1 + |\zeta_*|)^\gamma} d\zeta_* \right)^{\frac{1}{2}} \\ &\leq C \|f\|_{L_\zeta^*} \left(\int_{|\zeta - \zeta_*| \leq \frac{|\zeta|}{2}} \frac{e^{-\frac{1}{4}(|\zeta - \zeta_*|^2 + (\frac{|\zeta|^2 - |\zeta_*|^2}{|\zeta - \zeta_*|})^2)}}{|\zeta - \zeta_*|^2 (1 + |\zeta|)^{2(1-\gamma)} (1 + \frac{|\zeta|}{2})^\gamma} d\zeta_* \right. \\ &\quad \left. + \int_{|\zeta - \zeta_*| > \frac{|\zeta|}{2}} \frac{e^{-\frac{1}{4}|\zeta - \zeta_*|^2}}{|\zeta - \zeta_*|^2 (1 + |\zeta|)^{2(1-\gamma)}} d\zeta_* \right)^{\frac{1}{2}} \\ &\leq C \|f\|_{L_\zeta^*} \left[(1 + |\zeta|)^{-(3-\gamma)} + |\zeta|^{-2} (1 + |\zeta|)^{-(2-2\gamma)} \right]^{\frac{1}{2}}. \end{aligned}$$

Notice that $K(f)$ is also bounded. Therefore, we conclude the proposition. \square

We are ready to prove Lemma 5.2

proof of Lemma 5.2. We will use the spherical coordinate such that

$$(5.8) \quad \zeta' = (\rho \cos \theta, \rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi)$$

and also $r = s|\zeta|$ as we did in Section 2. We have

$$(5.9) \quad \begin{aligned} \|\tilde{f}(x, \cdot)\|_{L_\zeta^*}^2 &= \int_{\mathbb{R}^3} |\nu(\zeta)| \left| \int_0^{\tau_-(x, \zeta)} e^{-\nu s} K(f)(x - \zeta s, \zeta) ds \right|^2 d\zeta \\ &\leq \int_{\mathbb{R}^3} |\nu(\zeta)| \left| \int_0^{\tau_-(x, \zeta)} e^{-2\nu s} s^{-\alpha} ds \right| \left| \int_0^{\tau_-(x, \zeta)} |K(f)(x - \zeta s, \zeta)|^2 s^\alpha ds \right| d\zeta \\ &\leq C \int_{\mathbb{R}^3} (1 + |\zeta|)^\gamma \int_0^{\overline{p(x, \zeta)x}} \|f(x - r\hat{\zeta}, \cdot)\|_{L_\zeta^*}^2 (1 + |\zeta|)^{-(3-\gamma)} r^\alpha \frac{1}{|\zeta|^{1+\alpha}} dr d\zeta \\ &\leq C \int_0^\infty (1 + \rho)^{\gamma(1+\alpha)-3} \rho^{1-\alpha} \\ &\quad \cdot \int_0^\pi \int_0^{2\pi} \int_0^{\overline{p(x, \zeta)x}} \|f(x - r\hat{\zeta}, \cdot)\|_{L_\zeta^*}^2 r^\alpha dr \sin \theta d\phi d\theta d\rho \\ &\leq C \int_\Omega \frac{\|f(y, \cdot)\|_{L_\zeta^*}^2}{|x - y|^{2-\alpha}} dy \int_0^\infty (1 + \rho)^{\gamma(1+\alpha)-3} \rho^{1-\alpha} d\rho \\ &\leq C \int_\Omega \frac{\|f(y, \cdot)\|_{L_\zeta^*}^2}{|x - y|^{2-\alpha}} dy. \end{aligned}$$

Notice that the hard sphere case, $\gamma = 1$, is barely missed. We make an zero extension of f outside Ω , and, with abuse of notation, still call it f . Then, we conclude the lemma. \square

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